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Bernoulli's equation, energy and enthalpy

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In this paper, we expose and compare two different physical interpretations of Bernoulli's equation. One interpretation is that pressure must be a type of potential energy density, and Bernoulli's equation means the uniformity of the mechanical energy density in each streamline. However, the pressure behaves as potential energy density only for steady flows, a situation that includes Bernoulli's equation. Another interpretation is that pressure is not a type of potential energy density, the mechanical energy density changes along the streamlines, but the work from pressure balances the energy variation. The last interpretation implies that enthalpy density is uniform in each streamline. We conclude that the two interpretations are valid, it is not possible to exclude a particular interpretation, and both of them do not violate the law of conservation of energy.

Keywords: Bernoulli's equation, Fluid Dynamics, energy, enthalpy.

1. Introduction

Incompressible and inviscid fluids with a steady flow obey Bernoulli's equation (BE), which relates static pressure P, fluid velocity U, height z, gravitational acceleration g, density ρ, and stagnation pressure P_0 [1–3],

$$\frac{\rho U^2}{2} + \rho g z + P = P_0. \tag{1}$$

The constant P_0 is different for each streamline (see Figure 1). If the fluid is irrotational, P_0 is the same for every streamline [4] (see Figure 1). In this work, we will refer to static pressure as pressure.

A more popular expression of BE is

$$\frac{\rho U_1^2}{2} + \rho g z_1 + P_1 = \frac{\rho U_2^2}{2} + \rho g z_2 + P_2, \tag{2}$$

where the indices 1 and 2 represent two points in the same streamline (see Figure 1).

Although the mathematical solutions to BE are very simple, the physical interpretation of the equation can be very difficult, and the terminology is not unanimous [5–8]. For example, there is disagreement in the names of the terms $\rho U^2/2$, $\rho g h$, P and P_0 [8], controversy in the hypothesis in the derivations of BE [9], difficulties in the identification of the role of BE in lifting the plane's wings [10–12], problems in the microscopic interpretation of BE [4, 13, 14], questions about BE in different frames [15], etc. We have chosen for this article the problem of the physical interpretation of BE as conservation of energy or enthalpy density [4, 14, 16, 17].

If BE means the conservation of energy density in each streamline, pressure is a type of energy density (see the expression 1). The consequence of this physical interpretation must be the existence of pressure energy, a concept that has received much criticism [9, 16–18]. However, the pressure seems to behave as a type of potential energy density in BE [19]. The interpretation of BE as conservation of enthalpy density excludes the concept of pressure energy. A consequence of this interpretation is that the energy density is not constant in each streamline, but this would not violate the law of conservation of energy [4, 14]. We name these two interpretations "energy interpretation" and "enthalpy interpretation".

The goal of this work is to understand how there are two valid interpretations of BE. We intend to show the problems with each interpretation, but we will not look for reasons to exclude any point of view.

This article is organized as follows: Section 2 shows the conditions where the pressure behaves as a type of energy density; Sections 3 and 4 describe the "energy interpretation" and "enthalpy interpretation", respectively; and we present the discussion and conclusion in Sections 5 and 6.

2. Pressure as Type of Potential Energy Density

Usually, in Fluid Mechanics, pressure depends on position and time, $P = P(\vec{r}, t)$. Thus, there is a force density from the pressure gradient that depends on position and time, $\vec{f}p(\vec{r}, t) = -\nabla P(\vec{r}, t)$.

BE describes fluids in steady flow, so the fluid velocity and the pressure depend only on position, $U = U(\vec{r})$ and $P = P(\vec{r})$. We can rewrite BE in the expression (1) as

$$\frac{\rho U^2(\vec{r})}{2} + \rho g z + P(\vec{r}) = P_0. \tag{3}$$

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The potential energies depend on position, $E_{pot} = E_{pot}(\vec{r})$. Each conservative force is written as $F(\vec{r}) = -\nabla E_{pot}(\vec{r})$. An analogous relation is valid for conservative force densities. For example, the weight density $\vec{f}w(\vec{r}) = -\rho g\hat{k}$ and the gravitational potential energy density $\varepsilon_g(\vec{r}) = \rho gz$ obey the relation $\vec{f}w(\vec{r}) = -\nabla\varepsilon_g(\vec{r})$. Analogously, pressure force density is minus the pressure gradient $\vec{f}p(\vec{r}) = -\nabla P(\vec{r})$. Thus, pressure in BE behaves exactly as a type of potential energy density.

Curiously, the resultant force density in a fluid that is described by BE is $\vec{f} = -\rho g\hat{k} - \nabla P(\vec{r})$. This expression is the Navier-Stokes equation for steady flow without viscosity, the exact conditions of validity for BE.

In non-steady flows, pressure depends on time, $P = P(\vec{r}, t)$. Thus, pressure $P(\vec{r}, t)$ cannot be a type of pressure potential energy density in non-steady flows. Consequently, we cannot generalize the interpretation of pressure as a type of potential energy density.

We have reason to accept or deny the interpretation of pressure as a type of potential energy density in the context of BE. Each interpretation of pressure implies different physical meanings for BE.

3. Energy Interpretation

If we interpret pressure as a type of potential energy density, stagnation pressure P_0 in expression 1 is the total mechanical energy density in a point in a streamline. BE would represent that the mechanical energy density of an infinitesimal element of the fluid along a streamline is constant (see Figure 1). In fact, BE describes fluids without any type of dissipation or heat.

The interpretation of the pressure as potential energy density implies in the definition of potential pressure energy,

$$E_P = \int_R P(\vec{r})dV. \tag{4}$$

where R represents a region of the fluid.

We can illustrate the concept of pressure potential energy. In a stream tube, we choose four transversal surfaces, S_A, S_B, S_C and S_D (see Figure 2). Besides

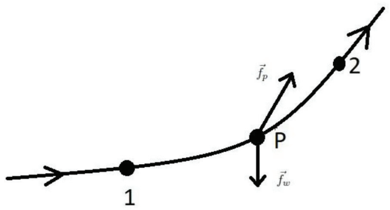

Figure 1: A fluid flows in a streamline. The points 1 and 2 are fixed in the same streamline. The point P represents an element of the fluid that moves along the streamline. The vectors represent the two force densities in the element of fluid, weight density $\vec{f}w$ and pressure force density $\vec{f}p$.

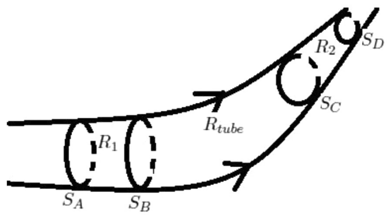

Figure 2: A fluid flows in a stream tube. The surfaces S_A, S_B, S_C and S_D define three regions in the stream tube: R_1, R_{tube} and R_2. The volumes of R_1 and R_2 are equal to ΔV. If there is pressure potential energy, the energy of an element of fluid that moves from R_1 to R_2 is conserved, and the energy in the region V_{tube} is constant. Alternatively, even if there is no pressure potential energy, the energy in the region V_{tube} is constant due to the balance of energy plus work from pressure, or enthalpy, in surfaces S_B and S_C.

that, we name the regions between S_A and S_B, S_B and S_C, and S_C and S_D respectively as R_1, R_{tube} and R_2 (see Figure 2). The volumes of regions R_1 and R_2 are equal to ΔV. Finally, the volume of region R_{tube} is $V_{tube} > \Delta V$. Thus, if the pressures in the regions R_1 and R_2 are uniform, the pressure energies 4 are respectively $P_1\Delta V$ and $P_2\Delta V$ (see Figure 2).

The fluid was in R_1 in the instant t_1 with energy E_1 (see Figure 2),

$$E_1 = \frac{(\rho\Delta V)U_1^2}{2} + (\rho\Delta V)gz_1 + P_1\Delta V.$$

The fluid left the region R_1, crossed R_{tube} and reached R_2 in the instant t_2 with the energy E_2,

$$E_2 = \frac{(\rho\Delta V)U_2^2}{2} + (\rho\Delta V)gz_2 + P_2\Delta V.$$

According to the BE in the expression 2, the energy of the element of fluid is constant, $E_1 = E_2$. We can interpret that the variations of kinetic and gravitational potential energies balance the potential pressure energy variation. BE means the conservation of the mechanical energy in any element of fluid during motion.

The fixed region R_{tube} receives and loses matter and energy through the surfaces S_B and S_C. A portion of fluid in region R_1 enters the region R_{tube} through S_B during a time interval Δt. The fluid transported the energy E_1 into the region R_{tube}. During the same time, Δt, the same volume of fluid leaves R_{tube} through S_C and occupies R_2. Thus, the region R_{tube} lost energy E_2. The identity $E_1 = E_2$ from BE in the expression 2 implies that the energy in the region R_{tube} is constant. We infer that the energy of any imaginary closed surface is constant.

4. Enthalpy Interpretation

If we do not interpret pressure as a type of potential energy density, the energy in the regions R_1 and R_2 are

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respectively

$$E_1 = \frac{(\rho\Delta V)U_1^2}{2} + (\rho\Delta V)gz_1,$$

and

$$E_2 = \frac{(\rho\Delta V)U_2^2}{2} + (\rho\Delta V)gz_2.$$

The BE in the expression 2 implies that $E_1 \neq E_2$. This would mean that the energy in V_{tube} changes with time. However, the fluid motion is associated with the work

$$W = \int_R PdV \tag{5}$$

We can interpret the pressure as a type of "work density". Although pressure is not a type of energy density, the work contributes to the energy. Thus, on the surface S_B, the region R_{tube} receives the energy from matter E_1 plus the work W_1,

$$E_1 + W_1 = \frac{\rho(\Delta V)U_1^2}{2} + \rho(\Delta V)gz_1 + P_1\Delta V.$$

Analogously, on the surface S_C, the region R_{tube} loses the energy E_1 from matter with the work W_2,

$$E_2 + W_2 = \frac{\rho(\Delta V)U_2^2}{2} + \rho(\Delta V)gz_2 + P_2\Delta V.$$

The energy plus the work is equal in the regions R_1 and R_2, $E_1 + W_1 = E_2 + W_2$, according to the BE in the expression 2. Thus, even if pressure is not a type of energy density, the energy in the region R_{tube} is constant.

In principle, the greatness $E + W$ would not be the enthalpy H itself because the definition is $H = E + PV$. However, the enthalpy in a volume ΔV must be $H = E + P\Delta V$. Thus, the conservation of energy density in R_{tube} results in the preservation of enthalpy in the regions R_1 and R_2.

Implicitly, the definition of enthalpy excludes the interpretation of pressure as energy density due to PV is not included in energy E, $H = E + PV$. Thus, in the expression 1, stagnation pressure P_0 means the enthalpy density. BE represents the conservation of enthalpy density in each streamline.

5. Discussion

The main contribution of this work is to compare two different interpretations of BE without trying to exclude one of them.

We can refute the refutations of the concept of energy density. The imaginary experiments that supposedly deny the existence of pressure energy [16–18] present pressure variations with time, a condition that excludes steady flows and BE in particular. Thus, these imaginary experiments do not invalidate the energy interpretation, but they confirm the restriction of the concept of potential pressure energy for steady flows [19].

In the enthalpy interpretation, the energy density is not uniform along a streamline. The fluids seem to violate the law of conservation of energy. The usual solution is that the law of conservation of energy relates the same energy to different instants, and BE connects different positions at the same instant [4, 14]. Thus, energy conservation does not imply uniform energy density. However, how can the fluid transport mass uniformly and energy non-uniformly? The present article contributes to this discussion because, even if pressure is not a type of energy density, the work from pressure adds or subtracts energy.

The concept of enthalpy is useful for describing isobaric transformations in ideal gases with uniform pressure. The work is $W = -P\Delta V$, where ΔV is the volume variation. The enthalpy variation is equal to heat, $\Delta H = Q$. Unlike isobaric transformations, BE describes inviscid incompressible fluids with null heat, $Q = 0$. The work is too $W = -P\Delta V$, but ΔV represents the volume of an element of incompressible fluid. The enthalpy variation is null, $\Delta H = 0$. We can apply the relation $\Delta H = Q$ in BE due to $Q = 0$, but there are deep differences between incompressible fluids in BE and ideal gases in isobaric transformations.

Apparently, we can reduce the two interpretations to a classification of pressure as "potential energy density" or "work density". However, the expressions 4 and 5 have different meanings. For non-steady flows, only the interpretation of pressure as work density is valid. Unlike the last situation, in fluids at rest without pressure variations, pressure behaves as a type of potential energy density, but there is no work from fluid motion. Thus, steady flows present a particular situation where there are two interpretations of pressure.

There is no experimental criterion to exclude one of the two interpretations. We can interpret an experimental pressure measurement as energy density or not.

For exemplifying BE with numerical values, Figure 3 illustrates five positions at the same streamline. Table 1 presents the values of kinetic energy density ($\rho U^2/2$), gravitational potential energy density (ρgh), and pressure (P) for each position of Figure 3. The fifth and sixth columns of Table 1 show the stagnation pressure (P_0)

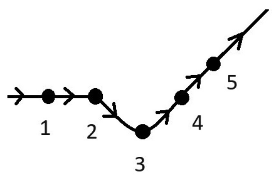

Figure 3: Positions 1, 2, 3, 4, and 5 are at the same streamline. The fluid motion follows the numerical sequence, and Table 1 shows the values of the kinetic and gravitational potential energy densities with pressure.

Table 1: From left to right, the columns present the five positions of Figure 1, kinetic energy density ($\rho U^2/2$), gravitational potential energy density (ρgh), pressure (P), stagnation pressure (P_0), and sum of the kinetic and gravitational potential energy densities ($\rho U^2/2 + \rho gh$) for each position. The unit of pressure and energy density is 10^5Pa.

Positions	$\rho U^2/2$ (10^5Pa)	Pgh (10^5Pa)	P (10^5Pa)	P_0 (10^5Pa)	$\rho U^2/2 + \rho gh$ (10^5Pa)
1	3	3	6	12	6
2	5	3	4	12	8
3	5	2	5	12	7
4	4	3	5	12	7
5	5	4	3	12	9

according to BE and the sum of kinetic and potential energy densities ($\rho U^2/2 + \rho gh$), respectively.

In the energy interpretation, the constant stagnation pressure in Table 1 is the total energy density. However, in the enthalpy interpretation, the total energy density must be the non-constant sum $\rho U^2/2 + \rho gh$. In this case, the pressure variation adds or subtracts the energy density $\rho U^2/2 + \rho gh$, and the stagnation pressure becomes the enthalpy density.

6. Conclusion

We conclude that we can interpret Bernoulli's equation in two ways: preservation of mechanical energy or enthalpy densities along each streamline. It is not possible to exclude a particular interpretation. Both interpretations are compatible with the law of conservation of energy.

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